Exercise 17

When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C.

(a) What is the temperature of the drink after 50 minutes?

(b) When will its temperature be 15°C?

Solution

Assume that the rate of decrease of the corpse's temperature is proportional to the difference between the corpse's temperature and the surrounding temperature T_s.

$$\frac{dT}{dt} \propto -(T - T_s)$$

The minus sign is included so that when the surroundings are cooler (hotter) than the corpse, dT/dt is negative (positive). Change this proportionality to an equation by introducing a positive constant k.

$$\frac{dT}{dt} = -k(T - T_s)$$

To solve this differential equation for T, make the substitution $y = T - T_s$.

$$\frac{dT}{dt} = -ky$$

Differentiate both sides of the substitution with respect to t to write the derivative in terms of y: $\frac{dy}{dt} = \frac{d}{dt}(T - T_s) = \frac{dT}{dt}$.

$$\frac{dy}{dt} = -ky$$

Divide both sides by y.

$$\frac{1}{y}\frac{dy}{dt} = -k$$

Rewrite the left side by using the chain rule.

$$\frac{d}{dt}\ln y = -k$$

The function you take a derivative of to get $-k$ is $-kt + C$, where C is any constant.

$$\ln y = -kt + C$$

Exponentiate both sides to get y.

$$e^{\ln y} = e^{-kt+C}$$

$$y = e^C e^{-kt}$$

Use a new constant A for e^C.

$$y(t) = Ae^{-kt}$$

Now that the differential equation has been solved, change back to the original variable T, the corpse's temperature.

$$T - T_s = Ae^{-kt}$$

As a result,

$$T(t) = T_s + Ae^{-kt}.$$

Since the room's temperature is 20°C, $T_s = 20$.

$$T(t) = 20 + Ae^{-kt}$$

Use the fact that the cup's initial temperature is 5°C.

$$5 = 20 + Ae^{-k(0)} \quad \rightarrow \quad A = 5 - 20 = -15$$

Consequently,

$$T(t) = 20 - 15e^{-kt}.$$

In order to determine k, use the fact that after 25 minutes the drink's temperature increases to 10°C.

$$10 = 20 - 15e^{-k(25)}$$

$$-10 = -15e^{-25k}$$

$$\frac{-10}{-15} = e^{-25k}$$

$$\frac{2}{3} = e^{-25k}$$

$$\ln\frac{2}{3} = \ln e^{-25k}$$

$$\ln\frac{2}{3} = (-25k)\ln e$$

$$k = -\frac{\ln\frac{2}{3}}{25} \approx 0.0162186 \text{ minute}^{-1}$$

Therefore, the drink's temperature after t minutes is

$$\begin{aligned} T(t) &= 20 - 15e^{-kt} \\ &= 20 - 15e^{-\left(-\frac{\ln\frac{2}{3}}{25}\right)t} \\ &= 20 - 15e^{\ln\left(\frac{2}{3}\right)^{t/25}} \\ &= 20 - 15\left(\frac{2}{3}\right)^{t/25}. \end{aligned}$$

Part (a)

Set $t = 50$ to get the drink's temperature after 50 minutes have passed.

$$T(50) = 20 - 15\left(\frac{2}{3}\right)^{50/25} = 20 - 15\left(\frac{2}{3}\right)^{2} = \frac{40}{3} \approx 13.3333$$

The drink's temperature after 50 minutes is about 13.3°C.

Part (b)

To find when its temperature is 15°C, set $T(t) = 15$ and solve the equation for t.

$$T(t) = 15$$

$$20 - 15\left(\frac{2}{3}\right)^{t/25} = 15$$

$$-15\left(\frac{2}{3}\right)^{t/25} = -5$$

$$\left(\frac{2}{3}\right)^{t/25} = \frac{1}{3}$$

$$\ln\left(\frac{2}{3}\right)^{t/25} = \ln\frac{1}{3}$$

$$\left(\frac{t}{25}\right)\ln\left(\frac{2}{3}\right) = \ln\frac{1}{3}$$

$$t = \frac{25\ln\frac{1}{3}}{\ln\frac{2}{3}} \approx 67.7378 \text{ minutes}$$